Coloring tournaments with few colors: Algorithms and complexity

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Abstract

A *k-coloring* of a tournament is a partition of its vertices into k acyclic sets. Deciding if a tournament is 2-colorable is NP-hard. A natural problem, akin to that of coloring a 3-colorable graph with few colors, is to color a 2-colorable tournament with few colors. This problem does not seem to have been addressed before, although it is a special case of coloring a 2-colorable 3-uniform hypergraph with few colors, which is a well-studied problem with super-constant lower bounds.

We present an efficient decomposition lemma for tournaments and show that it can be used to design polynomial-time algorithms to color various classes of tournaments with few colors, including an algorithm to color a 2-colorable tournament with ten colors. For the classes of tournaments considered, we complement our upper bounds with strengthened lower bounds, painting a comprehensive picture of the algorithmic and complexity aspects of coloring tournaments.

1 Introduction

A tournament $T = (V, A)$ is a complete, oriented graph: For each pair of vertices $i, j \in V$, there is either an arc from i to j or an arc from j to i (but not both). A subset of vertices $S \subseteq V$ induces the *subtournament* $T[S]$. If this subtournament contains no directed cycles, then it is said to be *acyclic*. The problem of *coloring a tournament* is that of partitioning the vertices into the minimum number of acyclic sets, sometimes referred to as the *dichromatic number* [Neu82]. Since a tournament contains a directed cycle if and only if it contains a directed triangle, the problem of coloring a tournament is equivalent to partitioning the vertices into the minimum number of sets so that each set does not contain a directed triangle.

Coloring tournaments can be compared to the problem of coloring undirected graphs. For the latter, deciding if a graph is 2-colorable (i.e., bipartite) is easy, but it is NP-hard to decide if a graph is 3-colorable. A widely-studied promise problem is that we are given a graph promised to be 3-colorable and the goal is to color it (in polynomial time) with few colors [Wig83, Blu94, KMS98, KT17]. For tournaments, it is easy to decide whether or not a tournament is 1-colorable (i.e., transitive), since this is exactly when the tournament is acyclic. However, deciding if a tournament is 2-colorable is already NP-hard [CHZ07].

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Graph Type	Lower Bound	Upper Bound
3-Colorable graphs	5 [BKO19], $O(1)^*$ [GS20]	$\tilde{O}(n^{0.19996})$[KT17]
k-Colorable graphs, $k \geq 3$	$2k-1$ [BKO19], $O(1)^*$[GS20]	$O(n^{1-\frac{3}{k+1}})$ [KMS98]
General graphs	$n^{1-\epsilon}$ [Has99, Zuc06]	$O(n(\log\log n)^2(\log n)^{-3})$ [Hal93]
3-Uniform 2-colorable hypergraphs	$O(1)$ [DRS05]	$\tilde{O}(n^{\frac{1}{5}})$ [KNS01]

Table 1: Best known lower and upper bounds for various graph coloring problems. All inapproximability results are under the assumption $\mathsf{P} \neq \mathsf{NP}$ except those denoted by *, which are under the d-To-1 Conjecture [Kho02]. The lower bound should be read as, "It is hard to color a 3-colorable graph with 5 colors." The upper bound as, "A 3-colorable graph can be (efficiently) colored with $\tilde{O}(n^{0.19996})$ colors."

This suggests the following promise problem: Given a tournament promised to be 2-colorable, what is the fewest number of colors with which it can be colored in polynomial time? This question is the starting point for this paper and naturally leads to related problems of determining upper and lower bounds for coloring various classes of tournaments. For comparison, the complexity landscape of graph coloring is well studied and we have a general understanding of what it looks like. (See Table 1.) In contrast, the problem of coloring tournaments has been studied very little from the algorithmic or complexity perspective. This paper is an effort to address this disparity.

Previous Work. The problem of coloring a 2-colorable tournament with few colors is a special case of coloring a 2-colorable 3-uniform hypergraph with few colors. Deciding if a 3-uniform hypergraph is 2-colorable is NP-hard [Lov73] and more recently it was proved to be NP-hard to color with any constant number of colors [DRS05]. On the positive side, a 2-colorable 3-uniform hypergraph can be colored in polynomial time with $\tilde{O}(n^{1/5})$ colors [AKMR96, CF96, KNS01], a result which uses tools from and is analogous to that of [KMS98] for 3-colorable graphs. Thus, $\tilde{O}(n^{1/5})$ is the best-known upper bound on the number of colors needed to efficiently color a 2-colorable tournament. Deciding if a tournament is 2-colorable is NP-hard [CHZ07] and furthermore, deciding if a tournament is k-colorable for any $k \geq 2$ is NP-hard [FGSY19]. It is consistent with these results that we can, say, efficiently color a 2-colorable tournament with three colors.

From a structural graph theory perspective, the problem of coloring tournaments has been widely studied due to its connection to the famous Erdős-Hajnal Conjecture [EH89, Chu14], which has an equivalent formulation in terms of tournaments [APS01]. The latter posits that for any tournament H, there is a constant ϵ_H (where $0 < \epsilon_H \leq 1$) such that any H-free tournament on n vertices has a transitive subtournament of size at least $O(n^{\epsilon_H})$. [BCC$^+$13] exactly characterize the tournaments for which $\epsilon_H = 1$, which they call *heroes*. Forbidding a hero in a tournament T actually results in T being colorable with a constant number of colors [BCC$^+$13], which yields a transitive induced subtournament of linear size. These results are existential and do not provide an efficient algorithm to color an H-free tournament with a constant number of colors, when H is some fixed hero.

Tournament Type	Lower Bound	Upper Bound
2-Colorable tournaments	2[CHZ07], 3	10
3-Colorable tournaments	5, $O(1)$ *	$\tilde{O}(n^{0.19996})$
k-Colorable tournaments, $k \geq 2$	$2k-1$, $O(1)$ *	$5 \cdot f(k-1) \cdot g(k)$
2-Colorable light tournaments	in P?	5
Light tournaments	in P?	8
General tournaments	$n^{\frac{1}{2}-\epsilon}$ †	$n/\log n$[EM64]

Table 2: Best known polynomial time inapproximability results and approximation algorithms for various tournament coloring problems. Previous results are indicated with a citation. All the results without a citation are established in this paper. Lower bounds are under the assumption $\mathsf{P} \neq \mathsf{NP}$ except those marked with a *, which hold under the d-To-1 Conjecture [Kho02]. The function $g(k)$ denotes the number of colors needed to efficiently color a k-colorable graph, while $f(k)$ is the number of colors needed to efficiently color a k-colorable tournament. The entry indicated by † is a hardness of approximation result.

Our Results. We consider some basic algorithmic and computational complexity questions on the subject of coloring tournaments. Our main algorithmic tool, presented in Section 2, is a decomposition lemma which can be used to obtain efficient algorithms for coloring tournaments in various cases when certain conditions are met. On a high level, it bears some resemblance to decompositions previously used to prove bounded dichromatic number in tournaments and in dense digraphs with forbidden subgraphs [BCC+13, HLNT19]. To apply our decomposition lemma to 2-colorable tournaments, we use an observation used by [AKMR96, CF96, KNS01] which states that there is an efficient algorithm to partition a 2-colorable tournament into two tournaments that are each light. A *light tournament* is one in which for each arc uv, the set of vertices $N(uv) = \{w \mid uvw \text{ forms a directed triangle}\}$ is transitive. (Let C_3 denote a directed triangle. A light tournament is H-free where H is the hero $(C_3, 1, 1)$.)

In fact, due to this observation and the fact that [BCC+13] showed that light tournaments have constant dichromatic number, it cannot be NP-hard (unless NP= co-NP) to color a 2-colorable tournament with $O(1)$ colors. (This does not however immediately imply that there is an efficient algorithm, since there are many search problems that are believed to be intractable even though their decision variant is easy, e.g., those in the class TFNP.) Although [BCC+13] did not provide an efficient algorithm to color a light tournament with a constant number of colors, a careful modification of their techniques indeed results in a polynomial-time algorithm using around 35 colors to color a light tournament. (This is discussed in Appendix C.)

Like some other lemmas which show that the dichromatic number of a tournament is bounded (i.e., constant) if the out-neighborhoods of vertices have bounded dichromatic number [HLTW19], our decomposition lemma also has a local-to-global flavor: If the sets $N(uv)$ can be efficiently colored with few colors for all arcs uv and if there are two vertices s and t such that the out-neighborhood of s and the in-neighborhood of t can be efficiently colored with few colors, then our decomposition lemma yields an efficient algorithm to color the whole tournament with few colors.

In Section 3, we give applications of our algorithmic decomposition lemma to color various classes of tournaments. Specifically, we show that 2-colorable tournaments can be efficiently colored with ten colors. We then use our toolbox to study 3-colorable tournaments. Here we show that the problem of coloring a 3-colorable tournament has a constant-factor reduction to the problem of coloring 3-colorable graphs. We also use our tools to show that light tournaments can be efficiently colored with eight colors, but since this is more technically involved than the other cases, we defer this to Section 5.

Next, we strengthen the lower bounds by showing in Section 4 that it is NP-hard to color a 2-colorable tournament with three colors. We then give a reduction from coloring graphs to coloring tournaments, which implies, for example, that it is hard to color 3-colorable tournaments with $O(1)$ colors under the d-To-1 Conjecture of Khot [Kho02]. Finally, we show that it is NP-hard to approximate the number of colors required for a general tournament to within a factor of $O(n^{1/2-\epsilon})$ for any $\epsilon > 0$. Our results are summarized in Table 2.

1.1 Notation and Preliminaries

Let $T = (V, A)$ be a tournament with vertex set V and arc set A. Sometimes, we use $V(T)$ to denote its vertex set and $A(T)$ to denote its arc set. For $S \subset V$, we use $T[S]$ to denote the subtournament induced on vertex set S, although we sometimes abuse notation and refer to the subtournament itself as S. We define $uv \in A$ to be an arc directed from u to v. We define $N^+(v)$ to be all $w \in V$ such that arc $vw \in A$ and $N^-(v)$ to be all $w \in V$ such that arc $wv \in A$. We let $N^+[v] = N^+(v) \cup \{v\}$ and $N^-[v] = N^-(v) \cup \{v\}$. For $S \subset V$, we define $N^+(S) = \bigcup_{v \in S} N^+(v)$, and we define $N^-(S), N^+[S], N^-[S]$ analogously. We use $N^{\pm}(S)$ to denote vertices in $V \setminus S$ that have at least one in-neighbor and at least one out-neighbor in S. Sometimes we refer to $N^{\pm}(S)$ of a set as its *mixed neighborhood*.

For $S, U \subset V$ such that $S \cap U = \emptyset$, we use $S \Rightarrow U$ to indicate that all arcs between S and U are directed from S to U. Let C_3 denote a directed triangle; usually, we refer to this simply as a triangle. Define $N(uv) \subset V$ to contain all vertices w such that uvw forms the directed triangle consisting of arcs uv, vw and wu. In other words, $N(uv) = N^-(u) \cap N^+(v)$. For three tournaments T_1, T_2 and T_3, we use $\Delta(T_1, T_2, T_3)$ to denote the tournament resulting from adding all arcs from T_1 to T_2, all arcs from T_2 to T_3 and all arcs from T_3 to T_1.

A tournament $T = (V, A)$ is *k-colorable* if there is a partition of V into k vertex-disjoint sets, $V_1, V_2, \ldots, V_k$, such that $T[V_i]$ is transitive for all $i \in \{1, \ldots, k\}$. We use $\vec{\chi}(T)$ to denote the *dichromatic number* of T (i.e., the minimum number of transitive subtournaments into which $V(T)$ can be partitioned). Computing the value $\vec{\chi}(T)$ is in general NP-hard [CHZ07]. We therefore use $\vec{\chi}_{\mathcal{C}}(T)$ to denote the number of colors by which T can be efficiently colored. Our goal is to find upper and lower bounds on $\vec{\chi}_{\mathcal{C}}(T)$. When the context is clear, we refer to the dichromatic number simply as the *chromatic number*.

We remark that we will always assume that a tournament T which we want to color is strongly connected; if this were not the case, we can color each strongly connected component separately. Therefore, each vertex has an out-neighborhood containing at least one vertex.

2 Efficient Tournament Decomposition for Coloring

We present a decomposition for a tournament that can be computed in polynomial time and yields an efficient method to color a tournaments with few colors in certain cases.

Definition 2.1. *We define a* c-vertex chain $(v_i)_{0\leq i\leq k}$ *of a tournament* T *the following way: Let* v_0 *and* v_k *be a pair of vertices such that* $\vec{\chi}_{\mathcal{C}}(N^+(v_0)\cup N^-(v_k))\leq c$*, and let* $(v_i)_{0\leq i\leq k}$ *be the vertices in the shortest directed path from* v_0 *to* v_k*.*

Additionally, we define an *arc chain* $(e_i)_{1\leq i\leq k}$ corresponding to a vertex chain, where e_i is the arc from v_{i-1} to v_i. The main idea behind this decomposition is to build zones that can be efficiently colored, and such that all arcs between zones at distance more than four (i.e., *long* arcs) go backwards.

Definition 2.2. *Given a c-vertex chain, a* path decomposition *of a tournament* T *is defined as:*

- $D_0 = N^+(v_0)$.
- *For* $1\leq i\leq k$, $D_i = N(e_i)\setminus(\cup_{0\leq j\leq i-1}D_j)$.
- $D_{k+1} = N^-(v_k)\setminus(\cup_{0\leq j\leq k}D_j)$.

First we prove that this is indeed a decomposition of T.

Lemma 2.3. *Let* $T=(V,A)$ *be a tournament and let* $(D_0,\ldots,D_{k+1})$ *be a path decomposition of* T*. Then* $V=\cup_{0\leq i\leq k+1}D_i$*.*

Proof. We will prove this lemma by contradiction: Suppose there is a vertex $w\in V$ that does not belong to any D_i. Assume that w does not belong to the vertex chain. Since w is neither in D_0 nor in D_{k+1}, then $w\in N^-(v_0)$ and $w\in N^+(v_k)$. Take the smallest integer i such that $w\in N^+(v_i)$. There must be one since $w\in N^+(v_k)$. Notice that $i\geq 1$ since $w\notin N^+(v_0)$, so e_i belongs to the arc chain and $w\in N(e_i)$. Therefore, $w\in D_i$, which is a contradiction.

Now consider the case in which w is in the vertex chain. An arc with both endpoints in the vertex chain that is not in the arc chain is backwards. Thus, $v_i\in N(e_{i+2})$ for all $0\leq i\leq k-2$. Notice that v_{k-1} can belong to D_{k+1} (if it does not belong to D_j for some $j<k+1$). Finally, $v_k\in N(e_{k-1})$. ☐

We remark that, for the sake of simplicity and to more easily visualize the decomposition, it might be easier to not include the vertices in the vertex chain in the path decomposition. In this case, these vertices can be colored with two extra colors. Since all arcs not in the arc chain with both endpoints in the vertex chain go backwards (with respect to the arc chain; otherwise there would be an even shorter path), we can use two colors so that all forwards arcs (those in the arc chain) are bicolored.

Lemma 2.4. *Let* $0\leq i,j\leq k+1$ *and let* $j\geq i+5$*. For* $u\in D_i$ *and* $w\in D_j$*, we have* $u\in N^+(w)$*.*

Proof. We will prove this by contradiction. Suppose $j \geq i+5$ and $u \in N^-(w)$. Then there is a path of three arcs from v_i to v_{j-1}, namely (v_i, u, w, v_{j-1}). (By definition of the decomposition, $u \in D_i$ implies $u \in N^+(v_i)$ and $w \in D_j$ implies $w \in N^-(v_{j-1})$.) This is not possible since by the definition of the vertex chain as the shortest path, there can be no path between v_i and v_{j-1} with fewer than four arcs (since $(j-1)-i \geq (i+5-1)-i = 4$). □

Lemma 2.5. *If T has a c-vertex chain that can be found in polynomial time and if $\vec{\chi}_\mathcal{C}(N(e)) \leq c$ for each arc e in the corresponding arc chain, then $\vec{\chi}_\mathcal{C}(T) \leq 5c$.*

Proof. Given a c-vertex chain, we construct a path decomposition. We make five palettes of c colors each with labels from 0 to 4. We color each D_i using the color palette with label $i \bmod 5$. Let us show that we can do this in polynomial time. First, note that the set of colors used is of size c for every D_i. Then, let us consider D_0: $N^+(v_0)$ can be colored efficiently with c colors by definition of a vertex chain. Similarly, D_{k+1} is a subset of $N^-(v_k)$ and can thus also be efficiently colored with c colors. Finally, for every $1 \leq i \leq k$, D_i is a subset of $N(e_i)$, which can be colored efficiently with c colors by the condition of the lemma.

Our goal is now to prove that this is a proper coloring of T. We will do this by showing that all forward arcs between different D_i are bicolored. By Lemma 2.4, there are no forwards arcs between D_i and D_j when $j \geq i+5$. Furthermore, by the definition of the coloring, no vertex in D_i and D_j can share a color for $i+1 \leq j \leq i+4$. Thus all forward arcs from D_i to D_j will be bicolored. Since every D_i is properly colored, and all forward arcs between different D_i are bicolored, T is properly colored. □

The next lemma has essentially the same proof as Lemma 2.5.

Lemma 2.6. *If T has a c-vertex chain that can be found in polynomial time and if $\vec{\chi}_\mathcal{C}(N(e)) \leq d$ for each arc e in the arc chain and if $c > d$, then $\vec{\chi}_\mathcal{C}(T) \leq c + 4d$.*

Proof. We find the path decomposition using the c-vertex chain. We can color the set $S = D_0 \cup D_{k+1}$ with c colors and the remaining sets D_i for $1 \leq i \leq k$ with d colors each. For the last $c-d$ of the colors used for S, we can remove these vertices from S since these colors will not be used again and call the remaining vertices in S (colored with the first d colors) S'. For the remaining vertices in S, we decompose them into $D_0 := D_0 \cap S'$ and $D_{k+1} := D_{k+1} \cap S'$ Now we have sets $D_0, D_1, \ldots, D_{k+1}$ each colored with d colors. We color these sets using five color palettes of d colors each and use the palette $i \bmod 5$ for set D_i. By Lemma 2.4, this does not create any monochromatic forward arcs. Thus, the total number of colors used is $(c-d) + 5d = c + 4d$. □

3 Algorithms for Coloring Tournaments

In this section, we consider the problems of coloring 2-colorable and 3-colorable tournaments, and we show how to use our tools to color them with few colors. We also consider the problem of coloring light tournaments, which is more technical and is therefore deferred to Section 5.

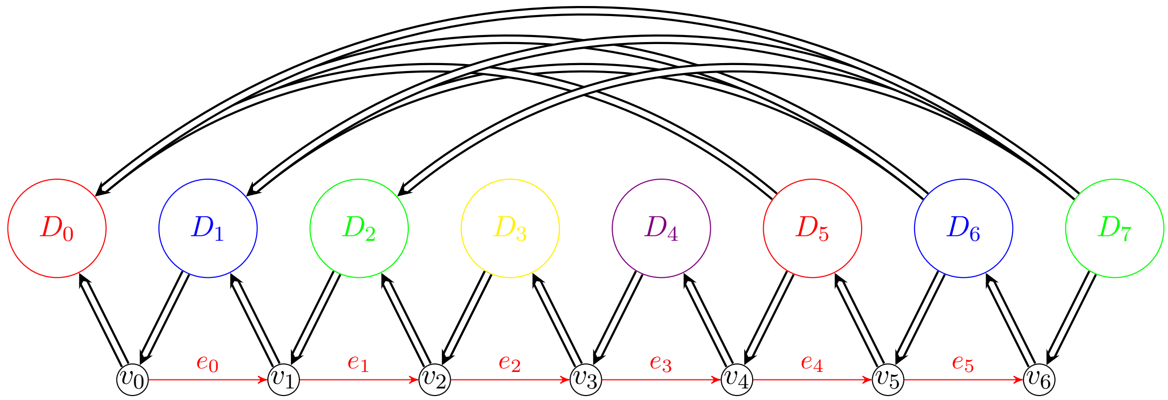

Figure 1: A path decomposition of T. The red arcs (e_i) form a shortest path from v_0 to v_k, thus all the arcs not depicted between the v_i's go backward. All the vertices in a given D_i are colored from the color palette indicated by the color of the D_i. Notice that because there are no long forward arcs between the D_i's, all arcs between D_i's that share a color palette are backwards.

3.1 2-Colorable Tournaments

A tournament $T = (V, A)$ is *2-colorable* if $\vec{\chi}(T) = 2$, and a 2-coloring of tournament T is a partition of V into two vertex sets, V_1 and V_2, such that $T[V_1]$ and $T[V_2]$ are each transitive. In this section, our goal is to prove Theorem 3.1.

Theorem 3.1. *Let T be a 2-colorable tournament. Then $\vec{\chi}_{\mathcal{C}}(T) \leq 10$.*

We say an arc uv in A is *heavy* if there exist three vertices $a, b, c \in N(uv)$ which form a triangle abc. If a tournament contains no heavy arcs, then it is *light*. We will use the following observation.

Observation 3.2. *Let T be a 2-colorable tournament. Then T can be partitioned into two light subtournaments T_1 and T_2 such that $\vec{\chi}_{\mathcal{C}}(T) \leq \vec{\chi}_{\mathcal{C}}(T_1) + \vec{\chi}_{\mathcal{C}}(T_2)$.*

This observation appears in [AKMR96, CF96, KNS01] where it is stated more generally for 2-colorable 3-uniform hypergraphs. We include a proof here for completeness.

Lemma 3.3. *In a 2-coloring of a tournament T, each heavy arc must be 2-colored.*

Proof. If u and v are both, say, blue, then each vertex in $N(uv)$ would be red, forcing a triangle in $N(uv)$ to be all red (i.e., monochromatic), which is not possible in a 2-coloring. □

Corollary 3.4. *In a 2-colorable tournament, the heavy arcs form a bipartite graph.*

Now we can prove Observation 3.2.

Proof of Observation 3.2. All heavy arcs can be easily detected. By Corollary 3.4, the set of heavy arcs forms a bipartite graph. The vertex set of this bipartite graph can be colored with two colors (red and blue), such that the tournament induced by each color does not contain a heavy arc. Then we partition the vertices into two sets one containing all the blue vertices and the other containing all the red vertices. The uncolored vertices can go in either set. Since neither of these sets contains any heavy arcs, we can partition the vertices of a 2-colorable tournament into two light subtournaments. □

Theorem 3.1 will follow from Observation 3.2 and the following theorem.

Theorem 3.5. *Let T be a 2-colorable light tournament. Then $\vec{\chi}_{\mathcal{C}}(T) \leq 5$.*

Our goal it to use Lemma 2.5 to prove Theorem 3.5. In other words, we want to show that a 2-colorable light tournament has a 1-vertex chain. We first prove a useful claim.

Lemma 3.6. *Let T be a k-colorable tournament. Then there exist vertices u and w such that $N^+(u) \cup N^-(w)$ is $(k-1)$-colorable.*

Proof. Since $T = (V, A)$ is k-colorable, there exist k transitive sets $X_1, \ldots, X_k$ such that $V = \cup_{i=1}^{k} X_i$. Then take u to be the vertex in X_1 that has only incoming arcs from other vertices in X_1 (i.e., the sink vertex for X_1). Similarly, take w to be the vertex in X_1 that has only outgoing arcs to other vertices in X_1 (i.e., the source vertex for X_1). The out-neighborhood of u and the in-neighborhood of w are both subsets of $V \setminus X_1$, and thus so is their union, which is therefore $(k-1)$-colorable. □

Now we are ready to prove that we can find a 1-vertex chain.

Lemma 3.7. *Let T be a 2-colorable, light tournament. Then T contains a 1-vertex chain that can be found in polynomial time.*

Proof. By Lemma 3.6, there exist u and w such that $N^+(u) \cup N^-(w)$ is transitive. To find them, we can test the transitivity of $N^+(u) \cup N^-(w)$ for every pair of vertices in T. Then we simply need to find a shortest path from u to w, which can be done in polynomial time. Let k denote the length of the path, and define $v_0 = u$, $v_k = w$, and $(v_i)_{1 \leq i \leq k-1}$ the rest of the vertices in the path. □

The proof of Theorem 3.5 follows from Lemma 3.7, Lemma 2.5 and the fact that $\vec{\chi}_{\mathcal{C}}(N(e)) \leq 1$ for every arc e in a light tournament.

Certificates of Non-2-Colorability In Section 3.1, we presented an algorithm to color a 2-colorable tournament with ten colors. Suppose we run this algorithm on an arbitrary tournament T (e.g., one that is *not* 2-colorable). Then our algorithm will either color T with ten colors or it will produce at least one certificate that T is not 2-colorable. A certificate will have the following form: either a) there is an odd cycle of heavy arcs in T, or b) for every ordered pair of vertices (u, v), the subtournament $T[N^+(u) \cup N^-(v)]$ is not transitive. In particular, an 11-chromatic tournament must contain such a certificate.

3.2 3-Colorable Tournaments

Coloring 3-colorable tournaments turns out to be closely related to coloring 3-colorable graphs. This seems surprising since the techniques for 3-colorable graphs were applied to coloring 2-colorable 3-uniform hypergraphs, which are a generalization of 2-colorable tournaments.

We will first show that we can adapt ideas of [Wig83] and [Blu94] to the problem of coloring 3-colorable tournaments by using our algorithm for coloring 2-colorable tournaments with ten colors as a subroutine.

Lemma 3.8. *A 3-colorable tournament can be colored with $O(\sqrt{n})$ colors in polynomial time.*

Proof. Let $T = (V, A)$ be a 3-colorable tournament. Notice that T has at least three vertices each of whose out-neighborhoods is 2-colorable. To see this, consider any proper 3-coloring of T. Each color spans a transitive subtournament and each transitive subtournament has a sink vertex that has outgoing arcs only towards the other two colors.

For any vertex, if its out-neighborhood is 2-colorable, we can color its out-neighborhood with 10 colors by Theorem 3.1. So we can try to run the algorithm for the out-neighborhood of every vertex, and the algorithm will successfully produce a 10-coloring of the out-neighborhood of at least three vertices.

Therefore, if the minimum outdegree is at least $\sqrt{n}$, we find a transitive set of size at least $\sqrt{n}/10$. On the other hand, if the minimum outdegree is smaller than $\sqrt{n}$, we will make progress another way. In this case, let u be a vertex with outdegree smaller than $\sqrt{n}$. Then, we add u to a set S, and continue the algorithm on the subtournament of T induced on $V \setminus N^+[u]$. We continue this until we find a transitive subtournament of size at least $\sqrt{n}/20$ or until we have removed half the vertices. In the first case, we will have found a transitive set of size $\Omega(\sqrt{n})$, and in the second case, the set S will be transitive, and also of size $\Omega(\sqrt{n})$.

In conclusion, since we can find a transitive set of size $\Omega(\sqrt{n})$ in polynomial time, we can repeat the procedure recursively to find a coloring with $O(\sqrt{n})$ colors in polynomial time (see [Blu94] for example). □

We can also use the decomposition of Section 2 to get a coloring with fewer colors based on a reduction to coloring 3-colorable graphs.

Theorem 3.9. *If we can efficiently color a 3-colorable graph G with k colors, then we can efficiently color a 3-colorable tournament with $50k$ colors.*

Proof. Let $T = (V, A)$ be a 3-colorable tournament. For every arc $e \in A$, try coloring $N(e)$ with 10 colors using Theorem 3.1. If the algorithm fails, the neighborhood of the edge is not 2-colorable, and thus the edge is not monochromatic in any 3-coloring. Let $F \subset E$ denote the set of arcs whose neighborhoods cannot be colored with 10 colors using our algorithm. Ignore the direction of the arcs in F and consider the graph $G = (V, F)$. This graph must be 3-colorable, since no arc in F is monochromatic in any 3-coloring of T.

Now let us show that from a coloring of G with k colors, we can obtain a coloring of T with $50k$ colors. Consider a coloring of the graph $G = (V, F)$ and let V_i be the vertices colored with color i

in this coloring. Consider the induced subtournament $T' = T[V_i]$; it has no arc in F and thus the neighborhood of every arc in this tournament can be colored efficiently with 10 colors. Furthermore, by Lemma 3.6 and Theorem 3.1, there are vertices u and v in T' such that $N^+_{T'}(u) \cup N^-_{T'}(v)$ is efficiently 10-colorable. So by Lemma 2.5, we can efficiently color T' with 50 colors. We can do this for the subtournament $T[V_i]$ for each of the i colors used to color G. $\square$

Combining this Lemma with approximation algorithm [KT17], which colors a 3-colorable graph with fewer than $n^{\frac{1}{5}}$ colors, we obtain the same asymptotic bound for 3-colorable tournaments.

Corollary 3.10. *Let T be a 3-colorable tournament on n vertices. Then, $\vec{\chi}_{\mathcal{C}}(T) \leq O(n^{0.19996})$.*

We can extend Theorem 3.9 to a more general case.

Lemma 3.11. *Let f and g be functions such that we can efficiently color k-colorable graphs (respectively, k-colorable tournaments) with $g(k)$ (respectively, $f(k)$) colors. Then $f(k) \leq 5 \cdot f(k-1) \cdot g(k)$.*

Proof. We use the same reduction as in the proof of Theorem 3.9, but now F is the set of arcs whose neighborhoods cannot be efficiently $f(k-1)$-colored. Then each V_i in G is colored with $5 \cdot f(k-1)$ colors. So we need a total of $5 \cdot f(k-1) \cdot g(k)$ colors. $\square$

4 Hardness of Approximate Coloring in Tournaments

In this section, we examine the hardness of approximate coloring of tournaments. [CHZ07] showed that deciding if a tournament can be 2-colored is NP-hard. For completeness, we provide a simplified (though similar) proof of this result in Appendix A. Later, [FGSY19] proved that for any k, it is NP-hard to decide if a tournament is k-colorable.

We will first improve upon these NP-hardness results and then show hardness of coloring k-colorable tournaments for $k \geq 3$ with $O(1)$ colors under the d-To-1 conjecture. The d-To-1 conjecture was first introduced by Khot alongside the famous Unique Games conjecture [Kho02], and has since been used to show hardness of coloring 3-colorable graphs with $O(1)$ colors [GS20].

First notice that the search problem must be at least as hard as its decision version.

Observation 4.1. *Let $k < \ell$ be any two constants. If we can color k-colorable tournaments with ℓ colors, then we can distinguish k-colorable tournaments from tournaments with chromatic number at least $\ell + 1$.*

This comes immediately from the fact that if we could ℓ-color all k-colorable tournaments, then we could see that they do not have chromatic number $\ell + 1$ or greater. The hardness of distinguishing between chromatic number k and greater or equal to $\ell + 1$ is therefore commonly established as a way of implying the hardness of coloring k-colorable graphs with ℓ colors (see for example [BKO19]).

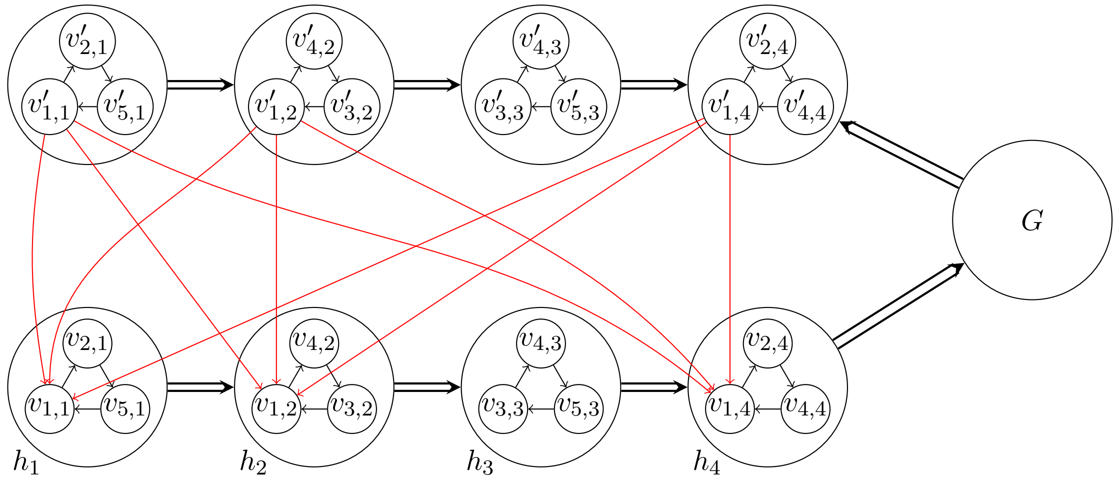

Figure 2: Construction of T from a 3-uniform hypergraph H. The edges in red (going down) were represented only for vertex v_1, but there is an arc from any vertex $v'_{a,i}$ towards all vertices $v_{a,j}$ for any j. The remaining arcs all go from the vertices $v_{a,i}$ towards the vertices $v_{b,j}$ for $a \neq b$ (they go up).

4.1 NP-Hardness of Approximate Coloring of k-Colorable Tournaments

It was shown previously that it is NP-hard to color a 2-colorable tournament with 2 colors [CHZ07, FGSY19]. We prove a stronger theorem, that it is NP-hard to 3-color a 2-colorable tournament.

Theorem 4.2. *For a tournament T, it is NP-hard to distinguish between the case in which $\vec{\chi}(T) = 2$ and the case in which $\vec{\chi}(T) \geq 4$.*

Proof. Let H be a 3-uniform hypergraph. In [FGSY19] and [CHZ07], they show how to construct a tournament G such that G is 2-colorable iff H is 2-colorable. We will build a new tournament $T = (V, A)$ such that if H is 2-colorable, T is also 2-colorable, and if H has chromatic number at least 7, then T has chromatic number at least 4. (Notice that it is NP-hard to color a 3-uniform 2-colorable hypergraph with c colors for any constant c [DRS05].)

We will start by defining a subtournament $T_1 = (V_1, A_1)$ of T. Given an enumeration of the hyperedges of H, $e_i = (v_a, v_b, v_c)$, we will add three vertices $v_{a,i}$, $v_{b,i}$ and $v_{c,i}$ to V_1, and add to A_1 the arcs $(v_{a,i}, v_{b,i})$, $(v_{b,i}, v_{c,i})$ and $(v_{c,i}, v_{a,i})$ such that these three vertices form a directed triangle. We then add the arcs from all the vertices $v_{a,i}$ towards all the vertices $v_{b,j}$ for any a, b, i, j with $i < j$. We make a copy of T_1, that we call $T_2 = (V_2, A_2)$, and add both to T. We then add the tournament G, and orient all arcs from vertices in V_1 towards vertices of G, and all arcs from vertices of G towards vertices in V_2. The only arcs we still need to orient are those between V_1 and V_2. For this, we look at the vertices of H from which the vertices of V_2 are derived; for $v_{a,i} \in V_1$ and $v'_{b,j} \in V_2$, we add an arc from $v'_{b,j}$ to $v_{a,i}$ iff $a = b$ (i.e., if they are derived from the same vertex of H), and we add an arc from $v_{a,i}$ to $v'_{b,j}$ otherwise. This completes the definition of T.

We will now establish that if H is 2-colorable, so is T. Given a 2-coloring of H, give all the vertices of V_1 the same color as the vertex of H they are derived from, and those in V_2 the opposite color of the vertex of H they are derived from. Finally color G with the same 2 colors. Then, any arc that goes from V_2 to V_1 will be 2-colored, and since all arcs are oriented from V_1 towards G and from G towards V_2, there can only be monochromatic triangles inside V_1, V_2 or G. However, G is properly 2-colored and thus does not have any monochromatic triangles. Furthermore, every triangle in V_1 and V_2 represents a hyperedge of H and must therefore contain two vertices of different colors.

It remains to show that if H has chromatic number at least 7, T has chromatic number at least 4. We will establish this by contradiction. Namely, we show that if T has a proper 3-coloring C, then we can construct a proper 6-coloring of H.

For every vertex v_a of H, consider the set of vertices $S_a = \{v_{a,i} \mid \forall i, v_{a,i} \in V_1\}$ and $Q_a = \{v_{a,i} \mid \forall i, v_{a,i} \in V_2\}$. A key property of our construction is that if H is not 2-colorable, then in any proper 3-coloring of T, either the set S_a or the set Q_a must be monochromatic. To see this, notice that if any vertex of S_a has the same color as any vertex of Q_a, then they will form a monochromatic triangle with a third vertex from G that has the same color (since G is colored with at least 3 colors). So if S_a and Q_a each use at least 2 out of 3 colors, then at least one color appears in both S_a and Q_a resulting in a monochromatic triangle.

Next we define a coloring C_H of H as follows. If S_a is monochromatic, then set $C_H(v_a) = C(S_a)$. Otherwise, if Q_a is monochromatic, then set $C_H(v_a) = C(Q_a) + 3$. Now take any hyperedge (v_a, v_b, v_c) of H; if the three sets S_a, S_b and S_c are monochromatic, then since there is a directed triangle $(v_{a,j}, v_{b,j}, v_{c,j})$ in T_1 for some j, the three vertices cannot have the same color in C, so they also do not all have the same color in C_H. If none of the three sets S_a, S_b and S_c are monochromatic, then the sets Q_a, Q_b and Q_c are each monochromatic, so the same argument applies. Finally, without loss of generality we can suppose S_a is monochromatic but not S_b. Then v_a and v_b do not have the same color in C_H by definition. Therefore, by case analysis, no hyperedge of H can be monochromatic, and thus C_H is a proper 6-coloring of H. □

Our goal is now to extend this hardness result to k-colorable tournaments. To do this, we will use an iterative construction presented in the following claims.

Claim 4.3. *Let a, b, c, d, e, ℓ be positive integers such that $e < c + d$. Let H be a 3-uniform hypergraph, and let R_1, R_2 and R_3 be three tournaments such that if $\chi(H) = 2$, then $\vec{\chi}(R_1) = a$, $\vec{\chi}(R_2) = b$ and $\vec{\chi}(R_3) = a + b$, and if $\chi(H) \geq \ell$, then $\vec{\chi}(R_1) \geq c$, $\vec{\chi}(R_2) \geq d$ and $\vec{\chi}(R_3) \geq e$.*

Then we can construct a tournament R' with chromatic number $\vec{\chi}(R') = a+b$ if $\chi(H) = 2$, and $\vec{\chi}(R') \geq e + 1$ if $\chi(H) \geq \ell$.

Proof. Let H be a hypergraph and let R_1, R_2 and R_3 be three tournaments that satisfy the conditions. Let $R' = \Delta(R_1, R_2, R_3)$. Now we want to show that if $\chi(H) = 2$, $\vec{\chi}(R') = a+b$. Simply color R_1 with a colors, R_2 with a new set of b colors, and R_3 with the same set of $a + b$ colors. These dicolorings will be proper since $\chi(H) = 2$. The dicoloring of R' is proper since there is no monochromatic triangle inside R_1, R_2 or R_3, and any triangle containing vertices from R_1 and R_2 will have at least two different colors.

Next we want to show that if $\chi(H) \geq \ell$, $\vec{\chi}(R') \geq e+1$. Suppose R' has a coloring with e colors. Then, since $c + d > e$, R_1 and R_2 must share at least one color. Furthermore, all e colors are used in R_3 by assumption. So there must be a monochromatic triangle since every triplet (u, v, w) with $u \in R_1$, $v \in R_2$, $w \in R_3$ forms a directed triangle. Thus, $\vec{\chi}(R') \geq e + 1$. □

Claim 4.4. *Let a, b, c, d, ℓ be positive integers. Let H be a 3-uniform hypergraph, and let R_1 and R_2 be two tournaments such that if $\chi(H) = 2$, then $\vec{\chi}(R_1) = a$ and $\vec{\chi}(R_2) = b$, and if $\chi(H) \geq \ell$, then $\vec{\chi}(R_1) \geq c$, $\vec{\chi}(R_2) \geq d$.*

Then there exists a tournament R' with $\vec{\chi}(R') = a+b$ if $\chi(H) = 2$ and $\vec{\chi}(R') \geq c+d$ if $\chi(H) \geq \ell$.

Proof. We will prove by induction on k with $a + b \leq k \leq c + d$, that there exists a tournament R'_k with $\vec{\chi}(R'_k) = a + b$ if $\chi(H) = 2$, and $\vec{\chi}(R'_k) \geq c + d$ if $\chi(H) \geq \ell$.

Initialization: For $k = a + b$, define R'_{a+b} to be any tournament with chromatic number $a + b$.

Induction: Suppose that for a fixed $k < c + d$, there is a tournament R'_k verifiying the conditions, then let us show that there is a tournament R'_{k+1} that verifies these same conditions for $k + 1$. We apply Claim 4.3 where R_1 and R_2 from both claims are the same, and R_3 is R'_k. This proves the existence of a tournament such that $\vec{\chi}(R'_k) = a + b$ if $\chi(H) = 2$, and $\vec{\chi}(R'_k) \geq c + d$ if $\chi(H) \geq \ell$.

Now we simply define $R' = R'_{c+d}$.

We remark that as every iteration of the construction can be done in polynomial time, and there are at most $c + d$ iterations, R' can be constructed in polynomial time and has size $|V(R')| \leq (c + d) \cdot (|V(R_1)| + |V(R_2)|) + |V(R_{a+b})|$. □

This gadget R' will allow us to prove that it is NP-hard to color a k-colorable tournament with $2k - 1$ colors.

Theorem 4.5. *For a tournament T, it is* NP*-hard to distinguish between the case in which $\vec{\chi}(T) = k$ and the case in which $\vec{\chi}(T) \geq 2k$.*

Proof. Given a 3-uniform hypergraph H, we will prove by strong induction on k that for every k, there exists a tournament T_k of size polynomial in $|V(H)|$ such that if $\chi(H) = 2$ then $\vec{\chi}(T_k) = k$, and if $\chi(H) \geq 7$ then $\vec{\chi}(T_k) \geq 2k$.

Initialization: For $k = 2$, we refer to the tournament constructed in the proof of Theorem 4.2.

For $k = 3$, let $T_3 = \Delta(T_2, T_2, T_2)$. If $\chi(H) = 2$, coloring the first copy with colors 1, 2, the second with colors 2 and 3, and the third with colors 3 and 1 yields a 3-coloring. This tournament is also not 2-colorable since in any 2-coloring, all copies of T_2 must use the same two colors, and thus there would be a monochromatic directed triangle.

If $\chi(H) \geq 7$, T_2 has chromatic number at least 4. Therefore, in any 5-coloring, two colors must be used in every copy of T_2, which would lead to a monochromatic directed triangle. Therefore, $\vec{\chi}(T_3) \geq 6$.

Induction hypothesis: For every $m \leq k$, there exists a tournament T_m of size polynomial in $|V(H)|$ such that if $\chi(H) = 2$, $\vec{\chi}(T_m) = m$, and if $\chi(H) \geq 7$, $\chi(T_m) \geq 2m$.

Induction: Let us show that there exists a tournament T_{k+1} of size polynomial in $|V(H)|$ such that if $\chi(H) = 2$, $\vec{\chi}(T) = k + 1$, and if $\chi(H) \geq 7$, $\vec{\chi}(G) \geq 2(k + 1)$.

Take the two tournaments $T_{\lfloor \frac{k+1}{2} \rfloor}$, $T_{\lceil \frac{k+1}{2} \rceil}$ obtained from the 3-uniform hypergraph H. These obey the conditions of Claim 4.4, thus there exists a tournament, that we call T_{k+1}, such that if $\chi(H) = 2$, $\vec{\chi}(T_{k+1}) = k + 1$, and if $\chi(H) \geq 7$, $\vec{\chi}(T_{k+1}) \geq 2(\lceil \frac{k+1}{2} \rceil + \lfloor \frac{k+1}{2} \rfloor) = 2(k + 1)$.

This concludes the induction. It immediately follows that it is NP-hard to distinguish between tournaments with chromatic number 2 and tournaments with chromatic number 4, as being able to do so would allow us to distinguish between 3-uniform hypergraphs with chromatic number 2, and 3-uniform hypergraphs with chromatic number at least 7, which is NP-hard [DRS05, Bha18]. □

4.2 Reduction from Coloring Graphs to Coloring Tournaments

In Section 3.2, we showed that if we can color a 3-colorable graph with k colors, then we can color a 3-colorable tournament with $50k$ colors. In this section, we give a reduction in the other direction. Specifically, we show that the problem of coloring a k-colorable graph with ℓ colors is reducible to the problem of coloring a k-colorable tournament with ℓ colors. A corollary of this reduction is hardness of coloring tournaments under the d-To-1 Conjecture of Khot [Kho02]; [GS20] showed that assuming the d-To-1 Conjecture, it is hard to color 3-colorable graphs with $O(1)$ colors, and using our reduction, we can extend this hardness to tournaments.

Theorem 4.6. *Given any two constants $k, \ell \geq 3$, if we can efficiently distinguish k-colorable tournaments and tournaments with chromatic number at least ℓ, then we can efficiently distinguish k-colorable graphs and graphs with chromatic number at least ℓ.*

We start by proving the following lemma that presents the building block of the reduction.

Lemma 4.7. *Let $G = (V_G, E_G)$ be a graph and $T = (V_T, A_T)$ a tournament such that $\vec{\chi}(T) = k$ when $\chi(G) = k$, and $\vec{\chi}(T) \geq min(\chi(G), c)$ when $\chi(G) > k$. We can build a new tournament $U = (V_U, A_U)$ such that $\vec{\chi}(U) = k$ when $\chi(G) = k$, and $\vec{\chi}(U) \geq min(\chi(G), c + 1)$ otherwise.*

Proof. Let $n_G = |V_G|$ and let $(T_i)_{1 \leq i \leq n_G - 1}$ be copies of T. Let $T_i = (V_i, A_i)$. Then $V_U := (\cup_{1 \leq i \leq n_G - 1} V_i) \cup V_G$. Fix an arbitrary ordering of the vertices in V_G. To build A_U, add the arc from v_j to v_i if $(v_i, v_j) \in E_G$, and the arc from v_i to v_j otherwise (i.e., if $(v_i, v_j) \notin E_G$). The resulting tournament induced on the vertices of V_G is said to have G as a *backedge graph*. Next we add all the arcs from v_i to all vertices of T_j for every $i \leq j$, and the arcs from every vertex of T_i to v_j for all $i < j$. Finally, we add the arcs from any vertex of T_i to any vertex of T_j for every $i < j$. This concludes the construction of U, which is depicted in Figure 3.

Suppose $\chi(G) = k$. Then let us show that $\vec{\chi}(U) = k$. In this case, $\vec{\chi}(T) = k$ by assumption. We take a k-coloring of G and a k-coloring of T and color the vertices in U (i.e., in V_G and in V_i for all $1 \leq i \leq n_G - 1$) according to these colorings. Notice that all arcs that are backwards with respect to the order $v_1 \rightarrow T_1 \rightarrow v_2 \rightarrow \ldots \rightarrow v_i \rightarrow T_i \rightarrow \ldots \rightarrow T_{|V_G|-1} \rightarrow v_{|V_G|}$ are bicolored. To see this, observe that arcs from v_j to v_i for $j > i$ belong to E_G and are therefore bicolored, and by construction, there are no arcs from v_j to T_i nor from T_j to T_i for $j > i$. Thus, there can only possibly be monochromatic triangles within T_i, but these sets are properly colored. Therefore, this is a proper dicoloring of the tournament U and $\vec{\chi}(U) = k$.

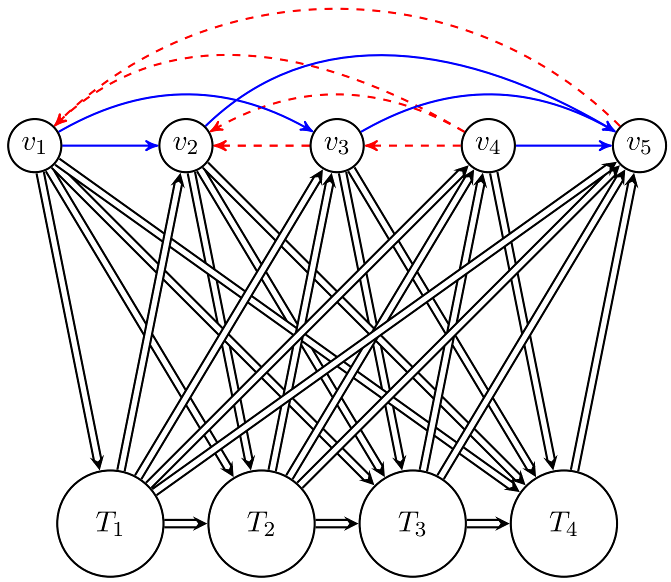

Figure 3: Construction of the tournament U from a graph G on five vertices. The dashed red edges are those present in G and all go backwards, whereas the remaining edges are blue and go forwards.

Let us now prove that when $\chi(G) > k$, we have $\vec{\chi}(U) \geq \min\{\chi(G), c+1\}$. By assumption, we have $\vec{\chi}(T) \geq \min\{\chi(G), c\}$ in this case. Thus, if $c = \chi(G)$, then the claim is true, since T is a subtournament of U. So let us consider the case in which $c < \chi(G)$. Then given a coloring of U with c colors, there must be a monochromatic edge (v_i, v_j) in G. Assuming without loss of generality that $i < j$, there is a monochromatic arc from v_j to v_i in U. Furthermore, since $\vec{\chi}(T) \geq c$, there must be some vertex of T_i that has the same color as v_i and v_j. Since all vertices in T_i form a directed triangle with v_i and v_j, this means that there is a monochromatic triangle in U, which is a contradiction. □

We can then prove Theorem 4.6 by a simple induction.

Proof of Theorem 4.6. Let $G = (V_G, E_G)$ be a graph and let $\ell \geq 3$ be a constant. For all $c \geq k$, let us build a tournament $T_c = (V_{T_c}, A_{T_c})$ by induction such that if $\chi(G) = k$, then $\vec{\chi}(T_c) = k$, and otherwise if $\chi(G) \geq \ell$, then $\vec{\chi}(T_c) \geq \min\{\chi(G), c\}$.

Initialization: For $c = k$, any k-colorable tournament satisfies the conditions.

Induction: Suppose there is a tournament T_c satisfying the conditions for a constant c. Let us show that there is a tournament T_{c+1} that satisfies these same conditions for $c+1$. This follows from Lemma 4.7 where T_c is T, and T_{c+1} is U.

Consider the tournament T_ℓ; it is of size $|V_{T_\ell}| = O(|V_G|^{2^\ell})$, which is polynomial for fixed ℓ. Furthermore, if $\chi(G) = k$ then $\vec{\chi}(T_\ell) = k$, and otherwise $\vec{\chi}(T_\ell) \geq \min\{\chi(G), \ell\}$, thus if we can efficiently decide if T_ℓ has chromatic number k or at least ℓ, we can also efficiently decide if G has chromatic number k or at least ℓ. ☐

Using the hardness of coloring 3-colorable graphs with a constant number of colors under the d-To-1 conjecture [GS20], we get equivalent hardness for coloring 3-colorable tournaments, and thus k-colorable tournaments for $k \geq 3$ (since any 3-colorable tournament is also k-colorable when $k \geq 3$).

Corollary 4.8. *Let $3 \leq k < \ell$ be any two constants. Then if the d-To-1 conjecture is true, we cannot distinguish between tournaments with chromatic number k and tournaments with chromatic number at least ℓ.*

Notice that if stronger hardness (for example constant hardness under the $\mathsf{P} \neq \mathsf{NP}$ assumption) were established for approximate coloring of 3-colorable graphs, then this reduction would provide stronger hardness results for 3-colorable tournaments (and thereby also for k-colorable tournaments). This would hold up to constant hardness, after which the blowup of the size of the tournament in the construction would be more then polynomial.

Finally, we consider the hardness of the problem of coloring general tournaments. Coloring digon-free digraphs has been shown to be NP-hard to approximate within a factor of $n^{1/2-\epsilon}$ [FHS19]. This proof can easily be extended to the case of tournaments, which provides the following theorem.

Theorem 4.9. *Given any arbitrarily small constant $\epsilon > 0$, it is* NP*-hard to approximate the chromatic number of tournaments within a factor of $n^{1/2-\epsilon}$.*

The proof of this Theorem is given in Appendix B.

5 Light Tournaments

Light tournaments are exactly those which do not contain the hero $\Delta(1, 1, C_3)$. [BCC+13] proved that light tournaments have constant chromatic number, but they did not state a precise constant, and their proof is not algorithmic. A careful modification of their approach can be used to give an algorithmic proof that this constant is around 35. Details are provided in Appendix C, since they could be useful in finding algorithms for tournaments with other forbidden heroes.

In this section, our goal is to prove the following theorem.

Theorem 5.1. *Let T be a light tournament. Then $\vec{\chi}_C(T) \leq 8$.*

Lemma 5.2. *Let T be a light tournament. Then we can find u, v such that:*

(i) $\vec{\chi}_C(N^+(u)) \leq 3$,

(ii) $\vec{\chi}_C(N^-(v)) \leq 3$, *and*

(iii) $\vec{\chi}_C(N^-(v) \cup N^+(u)) \leq 5$.

Assuming Lemma 5.2, we can prove Theorem 5.1.

Proof of Theorem 5.1. If the shortest path from u to v has length at least four, then notice that all arcs between $N^+(u)$ and $N^-(v)$ go from $N^-(v)$ to $N^+(u)$. Then by items (i) and (ii) from Lemma 5.2, we have $\vec{\chi}_{\mathcal{C}}(N^-(v) \cup N^+(u)) \leq 3$, so T has a 3-vertex chain. By Lemma 2.6, we can color T with seven colors.

Next, we consider the case in which the shortest path from u to v has length at most three. By item (iii) from Lemma 5.2, we have $\vec{\chi}_{\mathcal{C}}(N^-(v) \cup N^+(u)) \leq 5$. Moreover, each remaining vertex is in $N(e)$ for some edge e on the shortest path. So in total, we use at most eight colors. □

Now it remains to prove Lemma 5.2. We will start by establishing some structural claims about light tournaments which are adapted from [BCC+13]. For the rest of this section, $T = (V, A)$ will denote a light tournament. Note that we do not assume that T is necessarily 2-colorable. Recall that a C_3 is a directed triangle.

Definition 5.3. *Define a* C_3-chain *of length* ℓ *in* T *to be a set of* ℓ *vertex disjoint* C_3*'s,* $X = (X_1, X_2, X_3, \ldots, X_\ell)$*, such that for each* $i \in \{1, \ldots, \ell - 1\}$*,* $X_i \Rightarrow X_{i+1}$*.*

A *backwards arc* in a C_3-chain is an arc uv with $u \in X_i$ and $v \in X_j$ for $j < i$.

Lemma 5.4. *A* C_3*-chain has no backwards arcs.*

This follows from the following claim.

Claim 5.5. *If* $X = (X_1, X_2, \ldots, X_\ell)$ *is a* C_3*-chain of length* ℓ*, then* $X_i \Rightarrow X_j$ *for* $i < j$*, where* $1 \leq i < j \leq \ell$*.*

Proof. Notice that there are no arcs from X_{i+1} to X_i, since by definition of a C_3-chain, we have all arcs from X_i to X_{i+1}. Moreover, there is no arc uv from X_{i+2} to X_i since otherwise triangle X_{i+1} would appear in the neighborhood $N(uv)$, meaning that uv is heavy, which is a contradiction. This implies that all arcs go from X_i to X_{i+2} (since T is a tournament). Now suppose $j > i + 2$. If there is a back arc uv from $u \in X_j$ to $v \in X_i$, then uv is a heavy arc, because X_{j-1} would be in $N(uv)$ since by induction we have all arcs from X_i to X_{j-1} and from X_{j-1} to X_j. ◇

Let us fix $X = (X_1, X_2, \ldots, X_\ell)$ to be a C_3-chain in T, and let $W = V(T) \setminus V(X)$. Initially, X can be of any length $\ell \geq 1$.

Claim 5.6. *For* $w \in W$*:*

1. *If* $w \Rightarrow X_i$*, then* $w \Rightarrow X_j$ *for all* $j \geq i$*.*
2. *If* $X_i \Rightarrow w$*, then* $X_j \Rightarrow w$ *for all* $j \leq i$*.*

Proof. Suppose $w \Rightarrow X_i$ and there is an arc uw with $u \in X_j$ for $j > i$. Then uw is a heavy arc. Similarly, suppose $X_i \Rightarrow w$ and there is an arc wu with $u \in X_j$ for $j < i$, then wu is a heavy arc. ◇

We partition the vertices in W into zones $(Z_0, Z_1, \dots, Z_\ell)$ using the following criteria. For $w \in W$, if i is the highest index such that $X_i \Rightarrow w$, then w is assigned to zone Z_i. If there is no such X_i, then w is assigned to zone Z_0.

Say a vertex $w \in W$ is *clear* if $w \Rightarrow X_i$ or $X_i \Rightarrow w$ for all X_i in H. Let $C \subseteq W$ be the set of clear vertices.

Claim 5.7. *If C is not transitive, we can extend X.*

Proof. If the set $Z_i \cap C$ contains a triangle, then we can extend X by adding a new triangle to the chain between X_i and X_{i+1}.

If there is no i such that $Z_i \cap C$ contains a triangle, then we claim that C is transitive. This follows from the observation that there are no backwards arcs from $Z_j \cap C$ to $Z_i \cap C$ for $i < j$. Indeed, should such an arc uv from $Z_j \cap C$ to $Z_i \cap C$ exist, then $X_{i+1} \subset N(uv)$, so uv would be heavy. ◇

We say that X is a *maximal C_3-chain* if C is transitive. Let us also now define the *unclear* vertices U, where $U = W \setminus C$. In a maximal C_3-chain $X = (X_1, \dots, X_\ell)$, notice that for a vertex $a \in X_1$, we have $N^-(a) \cap U \subseteq N^\pm(X_1)$. (This is because if a vertex $u \in N^-(a)$ has $u \Rightarrow X_i$, then u would be a clear vertex.)

Claim 5.8. *We can efficiently find two directed triangles $X_1 = abc$ and $X_\ell = xyz$ such that the set $S = \{v \mid v \Rightarrow X_1 \text{ or } X_\ell \Rightarrow v\}$ is transitive.*

Proof. Find a maximal C_3-chain X and let ℓ be the length of this chain. Let $abc = X_1$ and $xyz = X_\ell$. The set of vertices $\{v \mid v \Rightarrow X_1 \text{ or } X_\ell \Rightarrow v\}$ is a subset of C and is therefore transitive. ◇

Claim 5.9. *Let xyz be a directed triangle. Then $\vec{\chi}_{\mathcal{C}}(N^\pm(\{x, y, z\})) \leq 3$.*

Proof. Each vertex $v \in N^\pm(\{x, y, z\})$ belongs to $N(xy), N(yz)$ or $N(zx)$. Since each of these sets is transitive, we conclude that $N^\pm(\{x, y, z\})$ can be colored with three colors. ◇

We can now prove Lemma 5.2.

Proof of Lemma 5.2. Recall that for a vertex $a \in X_1$, we have $N^-(a) \cap U \subseteq N^\pm(X_1)$. If $X_1 = abc$, notice that for $v \in N^-(a) \cap U$, $v \notin N(ca)$. Thus, $N^-(a) \cap U \subseteq N(ab) \cup N(bc)$, which is efficiently 2-colorable. Making an analogous argument for $X_\ell = xyz$ and $N^+(z) \cap U$, we conclude that $(N^+(z) \cup N^-(a)) \cap U$ is efficiently 4-colorable. The rest of the vertices in $N^+(z) \cup N^-(a)$ belong to the set S defined in Claim 5.8 and can be colored with one color. Therefore $\vec{\chi}_{\mathcal{C}}(N^+(z) \cup N^-(a)) \leq 5$. Moreover, we have $\vec{\chi}_{\mathcal{C}}(N^+(z)) \leq 3$ and $\vec{\chi}_{\mathcal{C}}(N^-(a)) \leq 3$. □

The approach in this section can be extended to bound the chromatic number of a more general subclass of heroes. See Appendix D for details.

It is a natural question to determine upper and lower bounds on the chromatic number of light tournaments (e.g., see Problem 1 in [MW11]). Theorem 5.1 gives an upper bound on the chromatic number of a light tournament. On the other hand, there exist light tournaments that

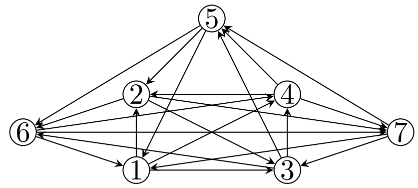

Figure 4: 3-chromatic light tournament.

are not 2-colorable. An example of such a tournament is the Paley tournament P_7, one of the four 3-chromatic tournaments on seven vertices [NL94]. This tournament is represented in Figure 4. We have not found any light tournament with chromatic number at least four. The Paley tournament P_{11} is the unique 4-chromatic tournament on 11 vertices [NL94]. A light 4-chromatic tournament would have to have at least 13 vertices as [BBKP22] proved that any 4-chromatic tournament on 12 vertices must contain an induced copy of P_{11} and P_{11} is not light.

Regarding the complexity of coloring a light tournament, notice that if we could show that it is hard to color a 2-colorable tournament with four colors (rather than three as per Theorem 4.2), this would imply hardness of coloring a 2-colorable light tournament with two colors by Observation 3.2. Indeed, we have no hardness results for coloring light tournaments. Any upper bound of c on their chromatic number would imply that it cannot be NP-hard to color them with c colors, because the property of being light is checkable in polynomial time (unlike the property of being, say, 2-colorable).

6 Conclusion

There are many open questions related to the theorems we have presented since all the rows in Table 2 present gaps between the upper and lower bounds. One example is light tournaments, which we discussed at the end of Section 5.

Another interesting topic is the relation of coloring tournaments and the feedback vertex set (FVS) problem on tournaments. There is an elegant 2-approximation for this problem [LMM$^+$21]. Notice that Theorem 3.1 implies that in a 2-colorable tournament, we can efficiently find a FVS of size at most $9n/10$. In contrast, the algorithm in [LMM$^+$21] could just return the whole vertex set if the two transitive sets were of roughly equal size. Analogous to a well-studied question for general graphs [DKPS10, KS14], one can ask what is the largest transitive induced subtournament that one can efficiently find in a 2-colorable tournament? Is it larger than $n/10$?

Finally, we remark that an implication of Theorem 3.9 is that proving any hardness of coloring 3-colorable tournaments would then provide hardness of coloring 3-colorable graphs with 50 times fewer colors. Since it has taken around 20 years to go from proving NP-hardness of coloring a 3-colorable graph with four colors [KLS00, GK00, GK04] to NP-hardness of coloring a 3-colorable graph with five colors [BKO19], it would be interesting to see if we can prove hardness of coloring 3-colorable tournaments for a constant larger than five (at least five is shown in Theorem 4.5), or perhaps show that the two problems are actually equivalent.

Acknowledgements

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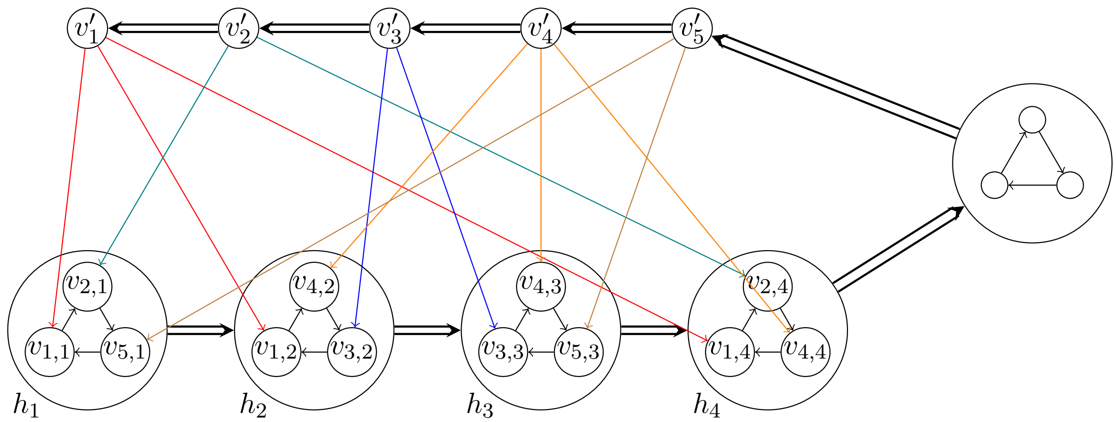

Figure 5: Construction of T from a 3-uniform hypergraph H. There is a downward arc between v'_b and all vertices $v_{b,i}$ for every b, i. These are the colored arcs in the figure. All remaining arcs all go from the vertices $v_{a,i}$ towards the vertices v'_b for $a \neq b$ (they go up).

A NP-Hardness of Deciding 2-Colorability

For completeness, we provide a proof of the NP-hardness of coloring 2-colorable tournaments with two colors. This proof is strongly inspired by the proof of [CHZ07].

Lemma A.1. *It is* NP*-hard to decide if a tournament has chromatic number two.*

Proof. We will reduce this problem to the problem of deciding 2-colorability of 3-uniform hypergraphs, which is known to be NP-hard [DRS05].

Let $H = (V_H, E_H)$ be a 3-uniform hypergraph. We now build a tournament $T = (V, A)$ such that T is 2-colorable iff H is 2-colorable.

We will start by defining a subtournament $T_1 = (V_1, A_1)$ of T. Given an enumeration of the hyperedges of H, $e_i = (v_a, v_b, v_c)$, we will add three vertices $v_{a,i}$, $v_{b,i}$ and $v_{c,i}$ to V_1, and add to A_1 the arcs $(v_{a,i}, v_{b,i})$, $(v_{b,i}, v_{c,i})$ and $(v_{c,i}, v_{a,i})$ such that these three vertices form a directed triangle. We then add the arcs from all the vertices $v_{a,i}$ towards all the vertices $v_{b,j}$ for any a, b, i, j with $i < j$. We now define a new subtournament $T_2 = (V_2, A_2)$ made up of three vertices that form a directed triangle. Finally, we define a last subtournament $T_3 = (V_3, A_3)$: $V_3 := V_H$, and T_3 forms a transitive set on its vertex set.

Then add T_1, T_2 and T_3 to T. Orient all arcs from vertices in V_1 towards vertices of V_2 and all arcs from vertices of V_2 towards vertices of V_3. The only arcs we still need to orient are those between V_1 and V_3. For this, we look at the vertices of H from which the vertices of V_1 and V_3 are derived; for $v_{a,i} \in V_1$ and $v'_b \in V_3$, we add an arc from v'_b to $v_{a,i}$ iff $a = b$ (i.e., if they are derived from the same vertex of H), and we add an arc from $v_{a,i}$ to v'_b otherwise. This completes the definition of T. Figure 5 gives an example of this construction for a hypergraph with five vertices and four hyperedges.

We will now establish that if H is 2-colorable, so is T. Given a 2-coloring of H, give all the vertices of V_1 the same color as the vertex of H they are derived from, and those in V_3 the opposite color of the vertex of H they are derived from. Finally color T_2 properly with the same 2 colors. Then, any arc that goes from V_3 to V_1 will be 2-colored, and since all arcs are oriented from V_1 towards V_2 and from V_2 towards V_3, there can only be monochromatic triangles inside V_1, V_2 or V_3. However, V_2 is a bicolored triangle, and, every triangle in V_1 and V_3 represents a hyperedge of H and must therefore contain two vertices of different colors.

It remains to show that if H has chromatic number at least 3, T has chromatic number at least 3. We will establish this by contradiction: We show that if T has a proper 2-coloring C, then we can construct a proper 2-coloring of H.

We define a coloring C_H of H by assigning to every vertex $v_a \in V_H$ the same color as its corresponding vertex $v'_a \in V_3$ has in C. Let us show that C_H is a proper 2-coloring of H. Notice that in a proper 2-coloring of G, $v_{a,i} \in V_1$ must have the opposite color of $v'_a \in V_3$, for any a, i. If it were not the case they would form a directed triangle with the vertex in V_2 of the same color, since T_2 is a directed triangle and must therefore be bicolored. Now suppose some hyperedge $e_i = (v_a, v_b, v_c)$ is monochromatic under C_H. Then $v'_a, v'_b, v'_c \in V_3$ all have the same color. Then, there is a triangle $(v_{a,i}, v_{b,i}, v_{c,i})$ in T_1 by definition, and all its vertices must have the same color (the opposite of that used for v'_a, v'_b, v'_c). This is a contradiction, thus all hyperedges of H are bicolored, and C_H is proper. □

B Hardness of Approximation for General Tournaments

In this section, our goal is to prove Theorem 4.9. Our proof parallels the proof of hardness of approximate coloring of digon-free digraphs of [FHS19]; we extend their approach to tournaments and show that it can be used to obtain hardness of approximation.

Theorem 4.9. *Given any arbitrarily small constant $\epsilon > 0$, it is* NP*-hard to approximate the chromatic number of tournaments within a factor of $n^{1/2-\epsilon}$.*

Lemma B.1. *Let ϵ be a constant such that $0 < \epsilon < 1$. There exists a tournament $T = (V, A)$ where $V = X + Y$ with $|X| = |Y| = n$ such that with probability going to 1 as n goes to infinity, for every two subsets $S_X \subseteq X$, $S_Y \subseteq Y$ having $|S_X| \geq n^\epsilon$, $|S_Y| \geq n^\epsilon$, the tournament induced by $S_X \cup S_Y$ contains a triangle.*

Proof. Define $T = (V, A)$ with $V = X \cup Y$ such that X and Y each form transitive tournaments on n vertices. Then orient all the remaining arcs randomly; so for $u \in X$, $v \in Y$, the arc goes from u to v with probability $1/2$.

Given $0 < \epsilon < 1$, take any $S_X \subseteq X$, $S_Y \subseteq Y$ with $|S_X| \geq n^\epsilon$ and $|S_Y| \geq n^\epsilon$. Take $u, v \in S_X$, and $w \in S_Y$. Then the probability of (u,v,w) forming a directed triangle in T is $1/4$, thus the probability of u and v forming no triangle with any vertex in S_Y is at most $(3/4)^{n^\epsilon}$ since the arcs between u and w and between v and w are oriented independently for every $w \in S_Y$. This probability tends to zero as n goes to infinity. □

The previous lemma can be derandomized using an explicit construction for bipartite Ramsey graphs [BRSW12].

Lemma B.2. *Let ϵ be a constant such that $0 < \epsilon < 1$. There exists a tournament $T = (V, E)$ where $V = X + Y$ with $|X| = |Y| = n$ such that for sufficiently large n, for every two subsets $S_X \subseteq X$, $S_Y \subseteq Y$ having $|S_X| \geq n^\epsilon$, $|S_Y| \geq n^\epsilon$, the tournament induced by $S_X \cup S_Y$ contains a triangle.*

Proof. Take a sufficiently large n. Then, from Theorem 1.3 in [BRSW12] there exists an explicit construction of a bipartite $o(n)$-Ramsey graph over n vertices. Let $B_1 = (X_1, Y_1, E_1)$ be such a graph. Then, define the tournament $T = (V, E)$ with $V = X + Y$ in the following way:

- $X = X_1$ and $Y = Y_1$
- Orient the arcs inside X and Y such that they both induce transitive tournaments.
- For every $u \in X$, $v \in Y$, orient the arc from u to v if $(u, v) \in E_1$ and from v to u otherwise.

Given $0 < \epsilon < 1$, take any $S_X \subseteq X$, $S_Y \subseteq Y$ with $|S_X| \geq n^\epsilon$ and $|S_Y| \geq n^\epsilon$. Let $x \in S_X$ and $y \in S_Y$ be the middle vertices of S_X and S_Y (ie. x has roughly equal in and out-degree in S_X, and y in S_Y). Without loss of generality, suppose that the arc between u and v is oriented from u to v. Then, the graph induced on B_1 by $S_X[n^-(u)]$ and $S_Y[n^+(v)]$ is of size at least $n^\epsilon - 2$, thus it is neither complete nor empty for sufficiently large n (since B_1 is a $o(n)$-Ramsey graph). This implies that there is an arc from a vertex $y \in S_Y[n^+(v)]$ to a vertex $x \in S_X[n^-(u)]$. Thus, there is a directed cycle (u, v, y, x) in $S_X \cup S_Y$, and since it is a tournament, there is some directed triangle. □

Theorem B.3. *It is* NP*-hard to find an acyclic induced subgraph of size greater than $n^{1/2+\epsilon}$ in an n^ϵ-colorable tournament, for every $0 < \epsilon < \frac{1}{4}$.*

Proof. For any $\epsilon > 0$, let $G = (V, E)$ be a graph on n_G vertices, colorable with n_G^ϵ colors. Feige and Kilian proved that it is NP-hard to find an independent set of size greater than n_G^ϵ in such graphs.

For each vertex $v_i \in V$, define a new transitive tournament T_i on n_G vertices. For each edge $(v_i, v_j) \in E$, join T_i and T_j such that they form the tournament of Lemma B.2, with T_i being X and T_j being Y. For all remaining $v_i, v_j \in V$ with $i < j$ (such that $(v_i, v_j) \notin E$), orient all arcs from each vertex of T_i to each vertex of T_j. This defines a new tournament T on $n = n_G^2$ vertices. T has an acyclic k-coloring with $k \leq n^{\epsilon/2}$ by coloring each T_i with the color of v_i in a $n_G^\epsilon = n^{\epsilon/2}$-coloring of G. Indeed, the only arcs that are not bicolored are inside a T_i for some i, or from a vertex of T_i to a vertex of T_j for $i < j$, and can thus never form a triangle. Let S be an acyclic induced subtournament of T. Notice that from Lemma B.2, if S intersects every $(T_i)_{i \in I}$ on more than n^ϵ vertices, then $(v_i)_{i \in I}$ forms an independent set of G. Therefore, if $|S| > 2n^{\frac{1+\epsilon}{2}}$, there must be at least n^ϵ tournaments that intersect S on at least n^ϵ vertices, which then leads to an independent set of size at least n^ϵ in G. □

The hardness of approximating a coloring in tournaments then comes as a corollary, as it immediately follows that it is NP-hard to distinguish a n^ϵ-colorable tournament from a k-colorable tournament with $k \geq n^{1/2-\epsilon}$, for every $0 < \epsilon < \frac{1}{4}$.

Corollary B.4. *Given any arbitrarily small constant $\epsilon > 0$, it is* NP*-hard to approximate the chromatic number of tournaments within a factor of $n^{1/2-\epsilon}$.*

C Light Tournaments

For the sake of completeness, we show that two other approaches from the literature can be adapted to obtain efficient algorithms for coloring light tournaments. We prove the following lemma, which is weaker than what we proved in Section 5.

Lemma C.1. *Let T be a light tournament. Then $\vec{\chi}_{\mathcal{C}}(T) \leq 35$.*

C.1 Algorithm I for Coloring Light Tournaments

Since a light tournament forbids a heavy edge, and a heavy edge is a hero (i.e., it is $\Delta(C_3, 1, 1)$), we show that the decomposition approach of [BCC+13] for bounding the dichromatic number of tournaments without a fixed hero can turned into an efficient algorithm to color a light tournament with approximately 35 colors. Throughout this section $T = (V, A)$ will denote a light tournament.

In Section 5, we already presented many of the necessary definitions. We consider a C_3-chain (Definition 5.3). Since there are no backwards arcs in a C_3-chain, we have the following corollary.

Corollary C.2. *A C_3-chain can be efficiently 2-colored.*

Let us fix $X = (X_1, X_2, \ldots, X_\ell)$ to be a maximal C_3-chain in T, and let $W = V(T) \setminus V(X)$. Recall that W is further partitioned into *clear* and *unclear* vertex sets denoted by C and U, respectively. A vertex v belongs to C if for every X_i, we have either $v \Rightarrow X_i$ or $X_i \Rightarrow v$. If C is transitive, then X is defined to be maximal. Notice that such a maximal C_3-chain can be found in polynomial time, while in the proof of [BCC+13], they used a maximum length C_3-chain (or more generally *jewel-chain*); it is not clear that a C_3-chain of maximum length can be found efficiently.

Now let us consider the unclear vertices U. Notice that if a vertex $u \in U$ belongs to zone Z_i for $i \in \{1, \ldots, \ell - 1\}$, then there is at least one arc from u to a vertex in X_{i+1}.

Claim C.3. *$\vec{\chi}_{\mathcal{C}}(Z_i \cap U) \leq 3$.*

Proof. If $z \in Z_i \cap U$, then $X_i \Rightarrow z$. However, we have $z \leftrightarrow X_{i+1}$. This means that z belongs to $N(uv)$ for some arc $uv \in X_{i+1}$. In other words, we can partition the vertices in $Z_i \cap U$ into three sets according to the three arcs in X_{i+1}. Since there are no heavy arcs, each of these three sets is transitive and we can color $Z_i \cap U$ with three colors. □

Claim C.4. *$\vec{\chi}_{\mathcal{C}}(X_i \cup (Z_i \cap U)) \leq 5$.*

Proof. We use two colors for X_i (which is a triangle) and three colors for $Z_i \cap U$. □

For simplicity, let us now assume that every vertex in $V(T)$ belongs to X or to U. Thus, we assume that $Z_i = Z_i \cap U$. (We only need one extra color for C since it is transitive.) Let $Y_0 = Z_0$ and for $i \in \{1, \ldots, \ell\}$, let $Y_i = X_i \cup Z_i$. Define $Y_i^L = \bigcup_{j=0}^{i-1} Y_j$ and $Y_i^R = \bigcup_{i+1}^{\ell} Y_j$.

Claim C.5. *Let $v \in Y_i$. Then,*

(i) $\vec{\chi}_{\mathcal{C}}(N^+(v) \cap Y_i^L) \leq 3$, and

(ii) $\vec{\chi}_{\mathcal{C}}(N^-(v) \cap Y_i^R) \leq 3$.

Proof. We consider several cases. The first case is $v \in X_i$. Notice that $N^+(v) \cap X_j = \emptyset$ for $j < i$, and $N^-(v) \cap X_j = \emptyset$ for $j > i$. We also note that $N^-(v) \cap (Z_{i+1} \cup \ldots \cup Z_\ell) = \emptyset$. Now observe $\vec{\chi}_{\mathcal{C}}(N^+(v) \cap (Z_0 \cup Z_1 \cup \ldots \cup Z_{i-1})) \leq 3$. This is because each $u \in N^+(v) \cap (Z_0 \cup Z_1 \cup \ldots \cup Z_{i-1})$ belongs to $N(xy)$ for some arc $xy \in X_i$.

Now consider the case where $v \in Z_i$. There are four subcases to consider.

1. $N^+(v) \cap (X_1 \cup \ldots \cup X_{i-1}) = \emptyset$.
2. $N^+(v) \cap (Z_0 \cup Z_1 \cup \ldots \cup Z_{i-1})$

   Consider $u \in N^+(v) \cap (Z_0 \cup Z_1 \ldots \cup Z_{i-1})$. If $u \Rightarrow X_i$, then vu is a heavy arc (since $X_i \subseteq N(vu)$). If $X_i \Rightarrow u$, then u would be in Z_i (at least). Thus, u belongs to $N(xy)$ for some arc $xy \in X_i$. Hence, $\vec{\chi}_{\mathcal{C}}(N^+(v) \cap (Z_0 \cup Z_1 \cup \ldots \cup Z_{i-1})) \leq 3$.
3. $N^-(v) \cap (X_{i+1} \cup \ldots \cup X_\ell)$.

   Since $\vec{\chi}_{\mathcal{C}}(X) \leq 2$, we have $\vec{\chi}_{\mathcal{C}}(N^-(v) \cap (X_{i+1} \cup \ldots \cup X_\ell)) \leq 2$.
4. $N^-(v) \cap (Z_{i+1} \cup \ldots \cup Z_\ell)$.

   Consider $x \in X_{i+1}$ such that arc vx is an arc. (Such an x exists, because it is not the case that $X_{i+1} \Rightarrow v$.) Now consider $u \in N^-(v) \cap (Z_{i+1} \cup \ldots \cup Z_\ell)$. We claim that $u \in N(vx)$. We conclude that $\vec{\chi}_{\mathcal{C}}(N^-(v) \cap (Z_{i+1} \cup \ldots \cup Z_\ell)) \leq 1$.

1. and 2. together imply (i) in the statement of the claim, and 3. and 4. imply (ii).

□

Lemma C.6. *Let $(Y_0, Y_1, \ldots, Y_\ell)$ be a partition of $V(T)$ such that $\vec{\chi}_{\mathcal{C}}(Y_i) \leq c_1$ and for each $v \in Y_i$:*

- $\vec{\chi}_{\mathcal{C}}(N^+(v) \cap (Y_0 \cup Y_1 \cup \ldots \cup Y_{i-1})) \leq c_2$, *and*
- $\vec{\chi}_{\mathcal{C}}(N^-(v) \cap (Y_{i+1} \cup \ldots \cup Y_\ell)) \leq c_2$.

Then $\vec{\chi}_{\mathcal{C}}(T) \leq 2(2c_1 + 2c_2 + 1)$.

Proof. Let $B \subset A(T)$ be the set of backwards arcs ($uv \in B$ if $u \in Y_i$ and $v \in Y_j$ for $j < i$). If there are no backwards arcs, then $\vec{\chi}_{\mathcal{C}}(T) \leq \max_{i \in \{0,1,\ldots,\ell\}}\{\vec{\chi}_{\mathcal{C}}(Y_i)\} \leq c_1$. Now we consider only a subset of backwards arcs chosen as follows: Choose the longest backwards arc u_1v_1 where $u_1 \in Y_\ell$. Suppose $v_1 \in Y_j$ for $j < \ell$. Let $T_1 = \bigcup_{i=j}^{\ell} Y_i$. Then choose the next backwards arc u_2v_2 with $u_2 \in V(T_1)$ and v_2 in Y_k for the smallest value of k possible, etc. Let $T_2 = \bigcup_{i=k}^{j-1} Y_i$, etc. Notice that if we consider the union of all T_i with odd i, there are no backwards arcs between them, and the same for T_i with even i. Suppose there are h such T_i's.

Then, $\vec{\chi}_{\mathcal{C}}(T) \leq 2 \cdot \max_{i \in \{1,\ldots,h\}} \vec{\chi}_{\mathcal{C}}(T_i)$. Now we claim that $\vec{\chi}_{\mathcal{C}}(T_i) \leq 2c_1 + 2c_2 + 1$. Consider the backwards arc uv, where $u \in Y_k$ and $v \in Y_j$ for $j < k$. We can color $N^-(v) \cap (T_i \setminus Y_j)$ and $N^+(u) \cap (T_i \setminus Y_k)$ each with c_2 colors. We can color Y_j and Y_k each with c_1 colors. Finally, we consider all vertices in $P = T_i \setminus \{Y_j \cup Y_k \cup N^+(u) \cup N^-(v)\}$. All vertices in P belong to $N(uv)$ and thus form a transitive tournament requiring one more color. □

So the algorithm to color a light tournament T is to find a maximal C_3-chain X. Next, color the clear vertices C with one color and remove C from T. Now consider the induced tournament on the remaining vertices and construct the partition $(Y_0, Y_1, \ldots, Y_\ell)$ based on X. Now follow the procedure in Lemma C.6.

Notice that $c_1 = 5$ and $c_2 = 3$. So Lemma C.6 uses 34 colors, and we add one more color to color C.

C.2 Algorithm II for Coloring Light Tournaments

[HLNT19] gave an algorithm to color a triangle-free dense digraph (with bounded independence number). We show how this approach can be adapted to give another algorithm to color a light tournament with a constant number of colors.

In this section, T is always a light tournament unless otherwise noted.

Definition C.7. *A set of vertices $B \subseteq V$ is a* bag *of T if for every triangle xyz in $V \setminus B$, there is some vertex $b \in B$ such that $\{x, y, z\} \Rightarrow b$ or $b \Rightarrow \{x, y, z\}$. Moreover, a bag must contain a directed triangle (i.e., it cannot be transitive).*

Observe that if B is not a bag of T, then we can color B with three colors. If B is a bag of T, then any S such that $B \subset S \subset V$ is also a bag of T. Also, note that V itself is a bag of T.

Claim C.8. *If $B \subset V$ is not a bag of T, then $\vec{\chi}_{\mathcal{C}}(T[B]) \leq 3$.*

Proof. If B is not a bag of T because it does not contain a triangle, then it is transitive. If it contains a triangle and is not a bag of T, then there is some triangle xyz such that $\{x, y, z\} \subset V \setminus B$ and for every $b \in B$, $b \in N^{\pm}(\{x, y, z\})$. Thus, we can apply Claim 5.9. ☐

We say a bag B is *poor* if for every triangle $xyz \in B$ either $N^{+}(\{x, y, z\})$ or $N^{-}(\{x, y, z\})$ is not a bag. We want to show that poor bags can also be colored with a constant number of colors.

Claim C.9. *If $B \subseteq V$ is a poor bag, then $\vec{\chi}_{\mathcal{C}}(T[B]) \leq 18$.*

Proof. Consider a poor bag B. Consider all triangles in B. For each triangle, either its in-neighborhood or its out-neighborhood is not a bag of T. If there is a triangle xyz in B such that both its in-neighborhood and its out-neighborhood are not bags of T, then we can color B with at most 11 colors: three for in-neighborhood, three for the out-neighborhood, two for the triangle and three for $N^{\pm}(\{x, y, z\})$.

So suppose for each triangle in B, its in-neighborhood is not a bag and its out-neighborhood is a bag, or vice-versa. According to these two possibilities, partition these triangles into L and R and consider the respective vertex sets (which can overlap). Consider R. These are triangles whose out-neighborhood is not a bag. Since there are no backwards arcs in a chain of C_3's, there must be some triangle xyz in R such that $N^{-}(\{x, y, z\}) \cap R$ does not contain a triangle. Thus, we can color its in-neighborhood with one color, its out-neighborhood with three colors, it's non-neighborhood with three colors, and the triangle itself with two colors, for a total of nine colors. We repeat the argument for L, so the maximum number of colors required is 18. ☐

Following [HLNT19], our plan is to find a chain of poor bags, put the remaining vertices into zones and show that there are no long backwards arcs between the zones.

We define $B = (B_1, B_2, \ldots, B_\ell)$ to be a *bag chain* of length ℓ if each B_i is a bag of T and $B_i \Rightarrow B_{i+1}$ for all $i \in \{1, 2, \ldots, \ell - 1\}$. Let $W = V(T) \setminus V(B)$. Assign $w \in W$ to zone Z_i if i is the highest index such that $B_i \Rightarrow w$.

Claim C.10. *Let $B = (B_1, B_2, \ldots, B_\ell)$ be a bag chain for a light tournament T. Let $(Z_0, Z_1, \ldots, Z_\ell)$ be a partition of $V(T) \setminus V(B)$ zones. The following properties hold:*

(a) $B_i \Rightarrow B_{i+r}$ *for every* $r \geq 1$,

(b) $B_i \Rightarrow Z_{i+r}$ *for every* $r \geq 0$,

(c) $Z_i \Rightarrow B_{i+r}$ *for every* $r \geq 3$,

(d) $Z_i \Rightarrow Z_{i+r}$ *for every* $r \geq 2$.

Proof. Property (a) holds for $r = 1$ by definition of a chain of bags. Now let $r \geq 2$. Suppose there is a backwards arc uv with $u \in B_{i+r}$ and $v \in B_i$. Since B_{i+1} contains a triangle, the arc uv is heavy, which is a contradiction.

By the partitioning criteria of vertices in $V(T) \setminus V(B)$ into zones, we have $B_i \Rightarrow Z_i$. If there is some arc uv with $u \in Z_i$ and $v \in B_j$ for $j < i$, then arc uv is heavy. Thus, $B_j \Rightarrow Z_i$ for all $j < i$.

To prove property (c), suppose there is an arc uv with $u \in B_{i+3}$ and $v \in Z_i$. Then there is some arc vx for $x \in B_{i+1}$ (otherwise, v would be in Z_{i+1}). Then uvx is a triangle. Since B_{i+2} is a bag of T, there is some vertex $y \in B_{i+2}$ such that $y \Rightarrow \{u, v, x\}$ or $\{u, v, x\} \Rightarrow y$. But this is not possible since $x \Rightarrow B_{i+2}$ and $B_{i+2} \Rightarrow u$. Thus, there is no such arc uv and we have $Z_i \Rightarrow B_{i+3}$. Now replace 3 with r.

To prove property (d), suppose that there is an arc uv with $u \in Z_{i+2}$ and $v \in Z_i$. Consider some $x \in B_{i+1}$ such that uxv is a triangle. Now since B_{i+2} is a bag of T, there is some $y \in B_{i+2}$ such that $y \Rightarrow \{u, x, v\}$ or $\{u, x, v\} \Rightarrow y$, which is a contradiction since $B_{i+1} \Rightarrow B_{i+2}$ and $B_{i+2} \Rightarrow Z_{i+2}$. Now replace 2 with $r \geq 2$. □

Now we need to show that we can color $B_i \cup Z_i$ efficiently with a constant number of colors. For this, we need the following observations.

Claim C.11. *A zone Z_i does not contain a bag chain of length at least five.*

Proof. If so, we can extend the principal bag chain B. □

Claim C.12. *A tournament without a bag chain of length five can be colored with c colors.*

Proof. Let $S \subset V$ be a set of vertices such that S does not contain a bag chain of length five for T. Either S itself is not a bag of T or S is a poor bag of T, in which case, we are done. Otherwise, we find a triangle in S and partition the remaining vertices according to the in- and out-neighborhoods of this triangle, and perhaps repeat this procedure to produce a bag chain of length at most four. Each vertex that is not in this bag chain is in the non-neighborhood of some triangle. There are at most three "pivot" triangles used. So in the end, the vertices of S are decomposed into a bag chain of at most four poor bags, a chain of C_3's, and five zones, each of which can be colored with three colors. So the number of colors required is at most $18 + 2 + 3 \cdot 3 = 29$. $\square$

D H_k-free tournaments

Definition D.1. *Let $\{H_k\}_{0 \le k}$ be the family of tournaments defined recursively with H_0 being a single vertex and $H_{k+1} = \Delta(H_k, 1, 1) \quad \forall k \ge 0$.*

Notice that $H_1 = C_3$. The set $\{H_k\}$ is a special class of *heroes*. Define f to be the function such that for any $k \ge 1$, $f(k) = \vec{\chi}_{\mathcal{C}}(T)$ where T is an H_k-free tournament. (In particular, $f(1) = 1$ and $f(2) \le 9$, as shown in Section 5.)

Lemma D.2. *For an integer $k \ge 0$, the number of colors needed to color an H_k-free tournament is $f(k) \le \prod_{i=1}^{k-1} 4i + 5$.*

The proof of this Lemma will follow the proof of Theorem 5.1: We will build a jewel-chain of H_k's in order to bound the chromatic number of the in-neighborhood of some vertex v and the out-neighborhood of another vertex u, which we will then use as the endpoints of a vertex chain. We start by extending the notion of heavy arcs to this setting.

Definition D.3. *We say an arc e is k-heavy if its neighborhood $N(e)$ contains an H_k.*

Notice that T is H_{k+1}-free iff it does not contain a k-heavy arc. For the rest of the section, let T be an H_{k+1}-free tournamant. Our goal will be to partition T into H_k-free sets, which will then allow us to color T by induction.

Definition D.4. *Define an H_k-*chain *of length ℓ in T to be a set of ℓ vertex disjoint H_k's, $X = (X_1, X_2, X_3, \ldots, X_\ell)$, such that for each $i \in \{1, \ldots, \ell - 1\}$, $X_i \Rightarrow X_{i+1}$.*

A *backwards arc* in a H_k-chain is an arc uv with $u \in X_i$ and $v \in X_j$ for $j < i$.

Lemma D.5. *An H_k-chain has no backwards arcs.*

This follows from the next claim.

Claim D.6. *Let T be an H_{k+1}-free tournament. If $X = (X_1, X_2, \ldots, X_\ell)$ is an H_k-chain of length ℓ, then $X_i \Rightarrow X_j$ for $i < j$, where $1 \le i < j \le \ell$.*

Proof. Notice that there are no arcs from X_{i+1} to X_i, since by definition of a H_k-chain, we have all arcs from X_i to X_{i+1}. Moreover, there is no arc uv from X_{i+2} to X_i since otherwise X_{i+1} would appear in the neighborhood $N(uv)$, meaning that $\{u\} \cup \{v\} \cup X_{i+1}$ forms an H_{k+1}, which is a contradiction. This implies that all arcs go from X_i to X_{i+2} (since T is a tournament). Now suppose $j > i + 2$. If there is a back arc uv from $u \in X_j$ to $v \in X_i$, then uv is a k-heavy arc, because X_{j-1} would be in $N(uv)$ since by induction we have all arcs from X_i to X_{j-1} and from X_{j-1} to X_j. $\square$

Let us fix $X = (X_1, X_2, \ldots, X_\ell)$ to be an H_k-chain in T, and let $W = V(T) \setminus V(X)$. Initially, X can be of any length $\ell \geq 1$.

Claim D.7. *For $w \in W$:*

1. *If $w \Rightarrow X_i$, then $w \Rightarrow X_j$ for all $j \geq i$.*
2. *If $X_i \Rightarrow w$, then $X_j \Rightarrow w$ for all $j \leq i$.*

Proof. Suppose $w \Rightarrow X_i$ and there is an arc uw with $u \in X_j$ for $j > i$. Then uw is a k-heavy arc. Similarly, suppose $X_i \Rightarrow w$ and there is an arc wu with $u \in X_j$ for $j < i$, then wu is a k-heavy arc. □

We partition the vertices in W into zones $(Z_0, Z_1, \ldots, Z_\ell)$ using the following criteria. For $w \in W$, if i is the highest index such that $X_i \Rightarrow w$, then w is assigned to zone Z_i. If there is no such X_i, then w is assigned to zone Z_0.

Say a vertex $w \in W$ is *clear* if $w \Rightarrow X_i$ or $X_i \Rightarrow w$ for all X_i in H. Let $C \subseteq W$ be the set of clear vertices.

Claim D.8. *If C is not H_k-free, we can extend X.*

Proof. If the set $Z_i \cap C$ contains an H_k, then we can extend X by adding a new H_k to the chain between X_i and X_{i+1}.

If there is no i such that $Z_i \cap C$ contains an H_k, then we claim that C is H_k-free. This follows from the observation that there are no backwards arcs from $Z_j \cap C$ to $Z_i \cap C$ for $i < j$. Indeed, should such an arc uv from $Z_j \cap C$ to $Z_i \cap C$ exist, then $X_{i+1} \subset N(uv)$, so uv would be k-heavy. Notice that since H_k is strongly connected, if an H_k were to belong to two different zones, it would create a backwards arc. Thus, we can conclude that if we cannot extend X, then C is H_k-free. □

We say that X is a *maximal H_k-chain* if C is H_k-free. Let us also now define the *unclear* vertices U, where $U = W \setminus C$. In a maximal H_k-chain $X = (X_1, \ldots, X_\ell)$, notice that for a vertex $a \in X_1$, we have $N^-(a) \cap U \subseteq N^o(X_1)$.

Claim D.9. *We can efficiently find two H_k's X_1 and X_ℓ such that the set $S = \{v \mid v \Rightarrow X_1$ or $X_\ell \Rightarrow v\}$ is H_k-free.*

Proof. Find a maximal H_k-chain X and let ℓ be the length of this chain. The set of vertices $\{v \mid v \Rightarrow X_1$ or $X_\ell \Rightarrow v\}$ is a subset of C and is therefore H_k-free. □

Claim D.10. *Let Y be an H_k. Then $\vec{\chi}_{\mathcal{C}}(N^o(Y)) \leq (2k+1) \cdot f(k)$.*

Proof. Take a Hamilton cycle $(e_i)_{1 \leq i \leq 2k+1}$ of Y. Each vertex $v \in N^o(Y)$ belongs to $N(e_i)$, for some i. Since each of these sets is H_k-free, we conclude that $N^o(Y)$ can be colored with $(2k+1) \cdot f(k)$ colors. □

We can now prove that H_{k+1}-free tournaments have bounded chromatic number by finding a $(4k+1) \cdot f(k)$-vertex chain.

Theorem D.11. *Let T be an H_{k+1}-free tournament. Then $\vec{\chi}_{\mathcal{C}}(T) \leq (4k+5) \cdot f(k)$.*

Proof. We just need to show that we can find a $(4k+1) \cdot f(k)$-vertex chain. Recall that for a vertex $a \in X_1$, we have $N^-(a) \cap U \subseteq N^o(X_1)$. If $a \in X_1$, and $(e_i)_{1 \leq i \leq 2k+1}$ is a Hamilton cycle of X_1 with $e_1 = ua$ and $e_2 = av$ for some vertices u and v, then notice that for $w \in N^-(a) \cap U$, $w \notin N(e_1)$. Thus, $N^-(a) \cap U \subseteq \cup_{2 \leq i \leq 2k+1} N^o(e_i)$, which is efficiently colorable with $2k \cdot f(k)$ colors, since it can be decomposed into $2k$ sets which are H_k-free and thus efficiently colorable with $f(k)$ colors. Making an analogous argument for $N^+(z) \cap U$, we conclude that $(N^+(z) \cup N^-(a)) \cap U$ is efficiently $4k \cdot f(k)$-colorable. The rest of the vertices in $N^+(z) \cup N^-(a)$ belong to the set S defined in Claim D.9 and can be colored with $f(k)$ colors. Therefore $\vec{\chi}_{\mathcal{C}}(N^+(z) \cup N^-(a)) \leq (4k+1) \cdot f(k)$, so we can use z and a as the endpoints of a $4k+1) \cdot f(k)$-vertex chain. Finally, it is clear that the neighborhood of an edge in an H_{k+1}-free tournament is H_k-free, and can thus be colored efficiently with $f(k)$ colors. Then we can apply Lemma 2.5 to prove that $\vec{\chi}_{\mathcal{C}}(T) \leq (4k+5) \cdot f(k)$. ☐

As an immediate corollary, we can bound the function f.

Corollary D.12. *For all integers k, the number of colors needed to color an H_k-free tournament $f(k) \leq \prod_{i=1}^{k-1} 4i + 5$.*